Personal notes: pressure wave and related

Yahoo!

This is part of a series of personal notes which may be littered with terrible typographical errors, scattered with gregarious grammatical mistakes or cluttered with catastrophic conceptual blunders.
A large part of this notes has input from the Hose! Couldnt have been possible without him!

Contents

1 Equations of state

2 Reynolds Transport Theorem

A useful theorem for formulating the various equations of continuum mechanics.
Let $\mathbf{f}(x,t)$ be a quantity of interest ($\mathbf{f}$ may be a tensor, vector, or scalar).
We are interested in calculating the time evolution of this quantity in a time-dependent region/volume, $\Omega(t)$, i.e.

$$\frac{d}{dt}\int_{\Omega(t)} \mathbf{f}(\mathbf{x},t)\,dV \tag{1}$$

where $\mathbf{x}$ is a point coordinate within $\Omega(t)$, and dV is a volume element at $\mathbf{x}$. Notice that it is a total derivative, not a partial one since the integrand is a function dependent solely on time (we would have already integrated $\mathbf{f}(\mathbf{x},t)$ over the entire time-dependent region before proceeding to differentiate it with respect to time).

We cannot simply move the time derivative into the integral. This is possible only if the limits of integration are constants, and independent of the variable, t. Then, we may consider the special case of

Leibniz's rule [1]:

$$\frac{d}{dt}\int_{\Omega} \mathbf{f}(\mathbf{x},t)\,dV = \int_{\Omega} \frac{d}{dt}\mathbf{f}(\mathbf{x},t)\,dV \tag{2}$$

However, $\Omega(t)$ is a constantly changing volume dependent on time. Its boundary $\partial\Omega$ is dynamic. So, one has to use the Reynolds transport theorem:

$$\frac{d}{dt}\int_{\Omega(t)} \mathbf{f}\,dV = \int_{\Omega(t)} \frac{\partial \mathbf{f}}{\partial t}\,dV + \int_{\partial\Omega(t)} \left(\mathbf{v}^b \cdot \mathbf{n}\right)\mathbf{f}\,dA \text{ (arbitrary volume)} \tag{3}$$

where $\mathbf{n}(\mathbf{x},t)$ is the outward-pointing unit normal vector, dA is the surface element at x, and $\mathbf{v}^b(\mathbf{x},t)$ is the velocity of this surface element. Note that $\mathbf{v}^b(\mathbf{x},t)$ is not the flow velocity since we are not assuming that $\Omega(t)$ is a material volume element. Fluid particles are free to enter or leave this arbitrarily defined volume element.

$$\frac{d}{dt}\int_{\Omega(t)} \mathbf{f}\,dV = \int_{\Omega(t)} \frac{\partial \mathbf{f}}{\partial t}\,dV + \int_{\partial\Omega(t)} (\mathbf{v} \cdot \mathbf{n})\,\mathbf{f}\,dA \text{ (material volume)} \tag{4}$$

If we consider that $\Omega(t)$ is made up of material elements, i.e. fluid parcels [2] which no material enters or leaves. Then the boundary elements will have to obey: $\mathbf{v}^b \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$. The velocity of the surface element, $\mathbf{v}^b$, has to be equal to the fluid velocity, $\mathbf{v}$, so that the flux across the dynamic boundary, $\partial\Omega(t) = 0$.

$$\frac{d}{dt}\int_{\Omega} \mathbf{f}\,dV = \int_{\Omega} \frac{\partial \mathbf{f}}{\partial t}\,dV \text{ (fixed volume)} \tag{5}$$

If we simplify further and consider that the volume is constant with respect to time, then $\mathbf{v}^b = 0$. Notice that this will be equivalent to apply the special case of Leibniz's integration rule, when the limits of integration is independent of the variable of integration.

3 Flux

q is a flow quantity of interest.
$\mathbf{u}$ is a velocity field describing the flow field in which q is following.
ρ is the volume density of q, i.e. quantity of q per unit volume.

So, we may now express $\mathbf{j}$ as the flux[3], defined as the amount of q flowing per unit time, through a unit area:

$$\mathbf{j} = \rho\mathbf{u} \tag{6}$$

Then, the amount of q passing through an arbitrary surface, S, per unit time is:

$$\iint_S \mathbf{j} \cdot d\mathbf{S} \tag{7}$$

This is the surface integral of flux over S.

[1] A special case of Leibniz's rule states that: $\frac{d}{dx}\left(\int_a^b \mathbf{f}(\mathbf{x},t)\,dt\right) = \int_a^b \frac{d}{dx}\mathbf{f}(\mathbf{x},t)\,dt$, where a and b are constants.

[2] A fluid parcel is a small volume of fluid particles. This volume of particles has dimensions much larger than the mean free path, λ, of individual fluid particles, yet is much smaller than the typical length scales, L of the specific flow under consideration, such that it adheres to the flow behaviour. So, naturally, it has a Kudsen number, $Kn = \frac{\lambda}{L} \ll 1$.

[3] there may be some ambiguity with the definition of flux and flux density, particularly so between transport phenomena and electromagnetism. Here, we shall define flux as the amount of quantity per unit time, per unit area.

4 Material derivative

Consider a material element subjected to a space-and-time dependent macroscopic velocity field, $\mathbf{u}$. Let $f(\mathbf{x}, t)$ be some physical quantity of the material element. f may be a tensor, vector or scalar-based field. The material derivative is then the time rate of change of f in such a velocity field, $\mathbf{u}$. Alternatively, one may also consider it as the time rate of change of f for a material element that is being transported with velocity $\mathbf{u}$ in this macroscopic velocity field (so we are assuming that the material element adheres strictly to the flow field velocity???)

$$\frac{Df}{Dt} \equiv \underbrace{\frac{\partial f}{\partial t}}_{\text{temporal}} + \underbrace{\mathbf{u} \cdot \nabla f}_{\text{spatial}} \tag{8}$$

The temporal term describes the intrinsic temporal variation of the field, independent of the presence of any flow (Eulerian-like description, akin to standing still at a point, observing the changes at that point). The spatial term contains the flow velocity describing the transport of the field in the flow (Lagrangian-like description). This gives the rate of change of f in response to its transport over the velocity field.

5 Conservation

Well, conservation laws are laws! You cannot destroy or create mass in a closed system (in a traditional sense... we do not go into more complex subjects like energy-mass equivalence etc). When describing the local conservation of a quantity, continuity is used. Either approach may be used: at a single point (differential) or a finite region (integral, be it area or volume).

Mass conservation (differential form):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = m \tag{9}$$

$$\underbrace{\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho}_{\text{material derivative}} + \rho \nabla \cdot \mathbf{u} = 0 \tag{10}$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0 \qquad \text{(normalised rate of change)} \tag{11}$$

where m is the source term. In most cases, $m = 0$. And if ρ is constant (independent of time and spatial dimensions), we may simplify further:

$$\nabla \cdot \mathbf{u} = 0 \tag{12}$$

From [1]:
Physically, Eqn. 11 states that if one follows a given (i.e., fixed mass) packet of fluid, the normalised time rate of change of its density is equal to the negative of the local divergence of the velocity. The local velocity divergence is, itself, directly proportional to the rate of change of volume of a fluid element; that is, its "dilatation rate," given by $\nabla \cdot \mathbf{u}$. Moreover, the dilatation rate is also equal to the instantaneous flux of fluid out of a differential volume element of space. This can be seen by integrating the dilatation rate over a volume and, utilising Gauss's divergence theorem:

$$\iiint_V \nabla \cdot \mathbf{u} \, dV = \iint_S \mathbf{u} \cdot \mathbf{n} \, dA = \iint_S \mathbf{u} \cdot d\mathbf{S} \tag{13}$$

The resulting surface integral is simply the instantaneous volume flux of fluid through the control surface. Compare this with flux: section 3.

Momentum conservation (differential form):
From [2]:

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) + \nabla\cdot\mathbf{P} = \mathbf{f} + m\mathbf{u} \tag{14}$$

$$\mathbf{u}\frac{\partial\rho}{\partial t} + \rho\frac{\partial\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u})\,\mathbf{u} + \rho\,(\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla\cdot\mathbf{P} = \mathbf{f} + m\mathbf{u} \tag{15}$$

$$\underbrace{\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right]}_{\text{mass conservation}}\mathbf{u} + \rho\underbrace{\left[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right]}_{\text{material derivative}} + \nabla\cdot\mathbf{P} = \mathbf{f} + m\mathbf{u} \tag{16}$$

$$m\mathbf{u} + \rho\frac{D\mathbf{u}}{Dt} + \nabla\cdot\mathbf{P} = \mathbf{f} + m\mathbf{u} \tag{17}$$

$$\rho\frac{D\mathbf{u}}{Dt} + \nabla\cdot\mathbf{P} = \mathbf{f} \tag{18}$$

where P is the fluid stress tensor given by $\mathbf{P} = p\mathbb{I} - \boldsymbol{\tau}$ or $P_{ij} = p\delta_{ij} - \tau_{ij}$, $\mathbf{f}$ is the external force density (like the gravitational force). It has the physical dimensions of force per unit volume. And the issuing mass adds momentum by an amount of $m\mathbf{u}$. In some cases, one can represent the effect of an object like a propeller by a force density f acting on the fluid as a source of momentum.

From [3]:

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P + \frac{\partial\tau_{i,j}}{\partial x_j}\mathbf{e}_i + \mathbf{f} \tag{19}$$

Or write it as in [1]:

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla P}{\rho} + \frac{\nabla\cdot\boldsymbol{\tau}}{\rho} + \sum_{i=1}^{N} Y_i\mathbf{F}_i \tag{20}$$

where $\mathbf{F}_i$ is a body force per unit mass acting on species i. It has the physical dimensions of force per unit mass, since $\mathbf{F} = \frac{\mathbf{f}}{\rho} = \frac{\text{force per unit vol}}{\text{density}}$. Y_i is the mass fraction of species i, and $\boldsymbol{\tau}$ is the viscous stress tensor given by:

$$\boldsymbol{\tau} = \mu_\lambda\,(\nabla\cdot\mathbf{u})\,\boldsymbol{\delta} + 2\mu\boldsymbol{S} \tag{21}$$

$$= \mu_\lambda\,(\nabla\cdot\mathbf{u})\,\boldsymbol{\delta} + 2\mu\frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right] \tag{22}$$

$\boldsymbol{\delta}$ is the Kronecker delta unit tensor, μ, μ_λ are the first and second coefficients of viscosity, and $\boldsymbol{S}$ is the symmetric strain rate tensor.

Ou selon le poly de l'Ecole Centrale Paris, on a:

$$\tau_{i,j} = -\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij} + 2\mu d_{ij} \tag{23}$$

$$= -\frac{2}{3}\mu\underbrace{\frac{\partial u_k}{\partial x_k}}_{\nabla\cdot\mathbf{u}}\delta_{ij} + 2\mu\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{24}$$

Note: we may also write $\frac{\partial u_i}{\partial x_j}$ as:

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{d_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\frac{\nabla\times\mathbf{u}}{2}} \tag{25}$$

From [1]:
The momentum equation is an expression of Newton's second law, stating that the net acceleration of a fixed fluid element ($\frac{D\mathbf{u}}{Dt}$) equals the force per unit mass exerted on it. The force terms on the right side denote surface forces due to pressure and viscous stress, and body forces.

Simplification (1):
If we consider a Newtonian fluid (constant μ) and incompressible fluid (constant $\rho \implies \nabla \cdot \mathbf{u} = 0$), the viscous stress tensor is reduced to:

$$\tau_{i,j} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \qquad \text{OR} \tag{26}$$

$$\boldsymbol{\tau} = \mu \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right] \tag{27}$$

Simplification (2):
If we consider a frictionless fluid, then $\nabla \cdot \boldsymbol{\tau}$ is negligible [3]:.

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \mathbf{f} \tag{28}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla P + \mathbf{f} \tag{29}$$

This is Euler's equation, which corresponds to the second law of Newton applied to a specific fluid element with a constant mass. The mass remains constant because we consider a specific material element. In the absence of friction, there are no tangential stresses acting on the surface of the fluid particle. The motion is induced by the normal stresses (pressure force) $-\nabla P$ and the bulk forces $\mathbf{F}$.

Simplification (3): If we consider the following:

- steady state, $\frac{\partial}{\partial t} = 0$
- non-viscous, $\mu = 0$, $\boldsymbol{\tau} = 0$
- body forces derive from a scalar potential, (i.e. conservative forces) $\mathbf{f} = -\rho \nabla \phi$[4]
- incompressible fluid, $\rho =$ constant

Then, we have the Bernoulli equation, written as:

$$\mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla P}{\rho} + \nabla \phi = 0 \tag{30}$$

$$\frac{\nabla (\mathbf{u} \cdot \mathbf{u})}{2} + \frac{\nabla P}{\rho} + \nabla \phi = 0 \tag{31}$$

$$\iiint \left(\frac{\nabla (\mathbf{u} \cdot \mathbf{u})}{2} + \frac{\nabla P}{\rho} + \nabla \phi \right) d\mathbf{x} = \text{constant} \tag{32}$$

$$\frac{\mathbf{u} \cdot \mathbf{u}}{2} + \frac{P}{\rho} + \phi = \text{constant} \tag{33}$$

Energy conservation (differential form): From [4]:
Perhaps the most complicated of the 3 conservation equations, since one may choose to express in terms of energy, enthalpy, or temperature. They are all equivalent. Note that because of continuity, one may use the material derivative. Eqn. 8 (which may be used in all the left hand sides of the enthalpy, energy, or temperature equations) holds for any quantity f.

$$\rho \frac{Df}{Dt} = \rho \left(\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f \right) \tag{34}$$

$$= \frac{\partial (\rho f)}{\partial t} + \frac{\partial (\rho u_i f)}{\partial x_i} \tag{35}$$

[4]example of a conservative force is gravitation: $\mathbf{g} = -\nabla \phi$. Friction is non-conservative.

We may write the total energy, e_t as:

$$\rho \frac{De_t}{Dt} = \frac{\partial \left(\rho e_t\right)}{\partial t} + \frac{\partial \left(\rho u_i e_t\right)}{\partial x_i} = -\frac{\partial q_i}{\partial x_i} \tag{36}$$

From [1]:

6 Arriving at the wave equation

Adapted from the poly de CERFACS...
Assume gas in duct:

- follows Navier-Stokes equations
- viscous effects are neglected
- 1-D flow: $P(\mathbf{x}, t) \rightarrow P(x, t)$, likewise for other variables
- no body force?

Furthermore, we consider the following:

- left tube contains fresh gas
- combustion occurs at the jump condition between the tubes
- right tube contains burnt gas
- isentropic flow in both tubes (i.e. no combustion)
  - This means that we may replace the energy equation by the isentropic relation

The 1D governing equations are then:
<u>mass conservation</u>

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u\right) = 0 \tag{37}$$

<u>momentum conservation</u>

$$\rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial P}{\partial x} \tag{38}$$

<u>energy conservation (isentropic)</u>[5]

$$\frac{P_2}{P_1} = \frac{\rho_2^\gamma}{\rho_1^\gamma} \tag{39}$$

$$\frac{P_2}{\rho_2^\gamma} = \frac{P_1}{\rho_1^\gamma} \tag{40}$$

$$P_2 = \frac{P_1}{\rho_1^\gamma}\rho_2^\gamma \tag{41}$$

$$P = C\rho^\gamma \tag{42}$$

where γ is the heat capacity ratio given by $\gamma = \frac{C_p}{C_v}$. C_p is heat capacity at constant pressure, and C_v is heat capacity at constant volume.

Then, we consider linear acoustics:

[5]this is a state equation, so logically the equation of a particular state, say state 1, is always expressed in term of another state 2

- small perturbations: $P = \overline{P} + P'$ with $P' \ll \overline{P}$
- likewise for $u = \overline{u} + u'$, $\rho = \overline{\rho} + \rho'$
- no mean flow: $\overline{u} = 0$

Re-writing the governing equations, linearising and neglecting all terms of 2nd order or higher. :
<u>mass conservation</u>

$$\frac{\partial\left(\overline{\rho}+\rho'\right)}{\partial t}+\frac{\partial}{\partial x}\left[\left(\overline{\rho}+\rho'\right)\left(\overline{u}+u'\right)\right]=0 \tag{43}$$

$$\frac{\partial\left(\overline{\rho}+\rho'\right)}{\partial t}+\frac{\partial}{\partial x}\left(\overline{\rho}\overline{u}+\overline{\rho}u'+\rho'\overline{u}+\rho'u'\right)=0 \tag{44}$$

$$\frac{\partial \rho'}{\partial t}+\overline{\rho}\frac{\partial u'}{\partial x}=0 \tag{45}$$

<u>momentum conservation</u>

$$\left(\overline{\rho}+\rho'\right)\left[\frac{\partial\left(\overline{u}+u'\right)}{\partial t}+\left(\overline{u}+u'\right)\frac{\partial\left(\overline{u}+u'\right)}{\partial x}\right]=-\frac{\partial\left(\overline{P}+P'\right)}{\partial x} \tag{46}$$

$$\overline{\rho}\frac{\partial u'}{\partial t}=-\frac{\partial P'}{\partial x} \tag{47}$$

<u>energy conservation (use series expansion):</u>

$$\frac{\overline{P}+P'}{\overline{P}}=\frac{\left(\overline{\rho}+\rho'\right)^{\gamma}}{\overline{\rho}^{\gamma}} \tag{48}$$

$$\overline{P}+P'=\overline{P}\overline{\rho}^{\gamma}\frac{\left(1+\frac{\rho'}{\overline{\rho}}\right)^{\gamma}}{\overline{\rho}^{\gamma}} \tag{49}$$

$$P'=\overline{P}\left(1+\frac{\rho'}{\overline{\rho}}\right)^{\gamma}-\overline{P} \tag{50}$$

$$P'=\overline{P}\left(1+\gamma\frac{\rho'}{\overline{\rho}}\right)-\overline{P} \tag{51}$$

$$P'=\gamma\overline{P}\frac{\rho'}{\overline{\rho}} \tag{52}$$

$$\rho'=\frac{P'}{\overline{P}}\frac{\overline{\rho}}{\gamma} \tag{53}$$

Now, we are only interested in pressure and velocity fluctuations (P', u'). Eliminate ρ' from Eqn. 45 using Eqn. 53:

$$\frac{\overline{\rho}}{\gamma\overline{P}}\frac{\partial P'}{\partial t}+\overline{\rho}\frac{\partial u'}{\partial x}=0 \tag{54}$$

$$\frac{\partial P'}{\partial t}+\overline{\rho}\frac{\gamma\overline{P}}{\overline{\rho}}\frac{\partial u'}{\partial x}=0 \tag{55}$$

Then, we end up with 2 equations and 2 unknowns (P', u'):

$$\frac{\partial P'}{\partial t}=-\overline{\rho}c^2\frac{\partial u'}{\partial x} \tag{56}$$

$$\overline{\rho}\frac{\partial u'}{\partial t}=-\frac{\partial P'}{\partial x} \tag{57}$$

where $c^2=\frac{\gamma\overline{P}}{\overline{\rho}}$

Notice that P' and u' are coupled. For any position x, if P' is time-invariant, then the spatial derivative of u' is zero. Likewise if u' is time-invariant, then the spatial derivative of P' is zero.

Note that when a fluctuating quantity (P' or u') is time-invariant, the fluctuation of that quantity is zero since the constant value of P' or u' will simply add to the temporal average, $\overline{P}$ and $\overline{u}$.

6.1 Decoupling pressure and velocity: approach 1

To arrive at the wave equation, take the temporal derivatives of Eqn. 56 and Eqn. 57:

$$\frac{\partial^2 P'}{\partial t^2} = -\overline{\rho}c^2 \frac{\partial^2 u'}{\partial t \partial x} \tag{58}$$

$$\overline{\rho}\frac{\partial^2 u'}{\partial t^2} = -\frac{\partial^2 P'}{\partial t \partial x} \tag{59}$$

And the spatial derivatives of Eqn. 56 and Eqn. 57:

$$\frac{\partial^2 P'}{\partial x \partial t} = -\overline{\rho}c^2 \frac{\partial^2 u'}{\partial x^2} \tag{60}$$

$$\overline{\rho}\frac{\partial^2 u'}{\partial x \partial t} = -\frac{\partial^2 P'}{\partial x^2} \tag{61}$$

We may then finally arrive at the wave equation by decoupling the pressure and velocity components. Eliminate the $\frac{\partial^2}{\partial t \partial x}$ terms by inserting Eqn. 61 into Eqn. 58, and Eqn. 60 into Eqn. 59.

$$\frac{\partial^2 P'}{\partial t^2} = c^2 \frac{\partial^2 P'}{\partial x^2} \tag{62}$$

$$\overline{\rho}\frac{\partial^2 u'}{\partial t^2} = \overline{\rho}c^2 \frac{\partial^2 u'}{\partial x^2} \tag{63}$$

Compare this with the 1-D homogeneous wave equation:

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2} \tag{64}$$

Pressure and velocity fluctuations follow wave equations! An isentropic condition leads to a homogeneous wave equation, since there is no heat addition. But of course, an isentropic condition is highly idealised. Not possible in real life. Also, recall that $c^2 = \frac{\gamma \overline{P}}{\overline{\rho}}$. This is the speed of propagation of waves.

6.2 Decoupling pressure and velocity: approach 2

In the above section, we saw how we may introduce double derivatives to decouple the pressure and velocity components, and arrive at the wave equation. Another way to decouple P' and u' is to consider matrix diagonalisation, to end up with linear equations!

Recall that we have:

$$\begin{aligned}
\frac{\partial P'}{\partial t} &= -\overline{\rho}c^2 \frac{\partial u'}{\partial x} \\
&= -\overline{\rho}\frac{\gamma \overline{P}}{\overline{\rho}} \frac{\partial u'}{\partial x} \\
&= -\gamma \overline{P} \frac{\partial u'}{\partial x} \\
\overline{\rho}\frac{\partial u'}{\partial t} &= -\frac{\partial P'}{\partial x}
\end{aligned}$$

where $c^2 = \frac{\gamma \overline{P}}{\overline{\rho}}$. We may then write it in matrix form:

$$\underbrace{\begin{bmatrix} P' \\ u' \end{bmatrix}_t}_{\mathbf{U}_t} + \underbrace{\begin{bmatrix} 0 & \gamma \overline{P} \\ \frac{1}{\overline{\rho}} & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} P' \\ u' \end{bmatrix}_x}_{\mathbf{U}_x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{0}$$

where the subscripts t and x denote differentiation with respect to t and x respectively. Then, we may diagonalise the matrix $\mathbf{A}$.

Recall how you will find eigenvalues and eigenvectors! Let λ be the eigenvalue, and $\mathbf{x}$ be the associated eigenvector. then we have:

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$
$$(\mathbf{A} - \lambda \mathbb{I})\,\mathbf{x} = 0$$

To arrive at a non-trivial solution, one would have to solve for:

$$\det(\mathbf{A} - \lambda \mathbb{I}) = 0$$

So, we have:

$$\det \begin{bmatrix} -\lambda & \gamma \overline{P} \\ \frac{1}{\overline{\rho}} & -\lambda \end{bmatrix} = 0 \qquad \text{(characteristic equation of } \mathbf{A}\text{)}$$
$$\Rightarrow \lambda^2 - \frac{\gamma \overline{P}}{\overline{\rho}} = 0 \qquad \text{(characteristic polynomial)}$$

This means that we have the eigenvalues:

$$\lambda = \pm \sqrt{\frac{\gamma \overline{P}}{\overline{\rho}}} = \pm c \qquad \left(\text{recall: } \frac{\gamma \overline{P}}{\overline{\rho}} = c^2\right)$$

Next, solve for the eigenvector when $\lambda = \pm c$:

$$\begin{bmatrix} 0 & \gamma \overline{P} \\ \frac{1}{\overline{\rho}} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \pm c \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

A quick observation that $x_1 = \pm \overline{\rho} c y_1$. So, the eigenvectors are $\begin{bmatrix} \pm \overline{\rho} c \\ 1 \end{bmatrix}$.
Notice later that you end up with $P' = \rho c (F - G)$, not quite the convention.
Since pressure is usually unaffected by other factors such as entropy waves, vorticity etc (while velocity is), it is better to keep the expression for P as clean and simple as possible. A better choice would be choosing eignvectors such that we end up with $P' = F + G$. So, from $x_1 = \pm \overline{\rho} c y_1$, you would want to choose eigenvectors: $\begin{bmatrix} 1 \\ \frac{1}{\overline{\rho} c} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -\frac{1}{\overline{\rho} c} \end{bmatrix}$.

We may then rewrite $\mathbf{A}$ as:

$$\begin{aligned}\mathbf{A} &= \mathbf{P\Lambda P^{-1}} \\ &= \begin{bmatrix}\overline{\rho}c & -\overline{\rho}c \\ 1 & 1\end{bmatrix}\begin{bmatrix}c & 0 \\ 0 & -c\end{bmatrix}\frac{1}{2\overline{\rho}c}\begin{bmatrix}1 & \overline{\rho}c \\ -1 & \overline{\rho}c\end{bmatrix} \\ &= \begin{bmatrix}\overline{\rho}c & -\overline{\rho}c \\ 1 & 1\end{bmatrix}\begin{bmatrix}c & 0 \\ 0 & -c\end{bmatrix}\begin{bmatrix}\frac{1}{2\overline{\rho}c} & \frac{1}{2} \\ -\frac{1}{2\overline{\rho}c} & \frac{1}{2}\end{bmatrix}\end{aligned}$$

Next we insert this expression into $\mathbf{U_t} + \mathbf{AU_x} = \mathbf{0}$.

$$\begin{aligned}\mathbf{U_t} + \mathbf{AU_x} &= \mathbf{0} \\ \mathbf{U_t} + \mathbf{P\Lambda P^{-1}U_x} &= \mathbf{0} \\ \mathbf{P^{-1}U_t} + \mathbf{\Lambda P^{-1}U_x} &= \mathbf{0} \qquad \text{(pre-multiply by } \mathbf{P}^{-1}\text{)} \\ \mathbf{V_t} + \mathbf{\Lambda V_x} &= \mathbf{0} \\ \begin{bmatrix}\mathcal{P} \\ \mathcal{U}\end{bmatrix}_t + \begin{bmatrix}c & 0 \\ 0 & -c\end{bmatrix}\begin{bmatrix}\mathcal{P} \\ \mathcal{U}\end{bmatrix}_x &= \mathbf{0}\end{aligned}$$

where $\mathbf{V} = \mathbf{P^{-1}U} = \begin{bmatrix}\mathcal{P} \\ \mathcal{U}\end{bmatrix}$.

So we have:

$$\begin{aligned}\frac{\partial \mathcal{P}}{\partial t} &= -c\frac{\partial \mathcal{P}}{\partial x} \\ \frac{\partial \mathcal{U}}{\partial t} &= c\frac{\partial \mathcal{U}}{\partial x}\end{aligned}$$

The vector $\mathbf{V}$ represents the Riemann invariants of the system (travelling waves) [5].

Also, notice the following:

$$\mathbf{PV} = \begin{bmatrix}\overline{\rho}c & -\overline{\rho}c \\ 1 & 1\end{bmatrix}\begin{bmatrix}\mathcal{P} \\ \mathcal{U}\end{bmatrix} = \begin{bmatrix}P' \\ u'\end{bmatrix} = \mathbf{U}$$

Then, we would have:

$$\begin{aligned}P' &= \overline{\rho c}\,(\mathcal{P} - \mathcal{U}) \\ u' &= \mathcal{P} + \mathcal{U}\end{aligned}$$

which leads to:

$$\begin{aligned}P' &= f\left(t - \frac{x}{c}\right) - g\left(t + \frac{x}{c}\right) \\ u' &= f\left(t - \frac{x}{c}\right) + g\left(t + \frac{x}{c}\right)\end{aligned}$$

Looks a bit different? That's due to the choice of the eigenvectors! As mentioned earlier, ideally, you would want to choose eignevectors such that we end up with $P' = F + G$.

And if you had used $\begin{bmatrix} 1 \\ \pm\frac{1}{\bar{\rho}c} \end{bmatrix}$ s the eigenvectors, we would have:

$$
\begin{aligned}
\mathbf{A} &= \mathbf{P\Lambda P^{-1}} \\
&= \begin{bmatrix} 1 & 1 \\ \frac{1}{\bar{\rho}c} & -\frac{1}{\bar{\rho}c} \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix} \left(-\frac{\bar{\rho}c}{2}\right) \begin{bmatrix} -\frac{1}{\bar{\rho}c} & -1 \\ -\frac{1}{\bar{\rho}c} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ \frac{1}{\bar{\rho}c} & -\frac{1}{\bar{\rho}c} \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\bar{\rho}c}{2} \\ \frac{1}{2} & -\frac{\bar{\rho}c}{2} \end{bmatrix}
\end{aligned}
$$

The vector $\mathbf{U}$ would be:

$$
\mathbf{PV} = \begin{bmatrix} 1 & 1 \\ \frac{1}{\bar{\rho}c} & -\frac{1}{\bar{\rho}c} \end{bmatrix} \begin{bmatrix} \mathcal{P} \\ \mathcal{U} \end{bmatrix} = \begin{bmatrix} P' \\ u' \end{bmatrix} = \mathbf{U}
$$

Then, we would have:

$$
\begin{aligned}
P' &= \mathcal{P} + \mathcal{U} \\
u' &= \frac{1}{\bar{\rho}c}\left(\mathcal{P} - \mathcal{U}\right)
\end{aligned}
$$

This would be in a form that one is familiar with:

$$
\begin{aligned}
P' &= f\left(t - \frac{x}{c}\right) + g\left(t + \frac{x}{c}\right) \\
u' &= f\left(t - \frac{x}{c}\right) - g\left(t + \frac{x}{c}\right)
\end{aligned}
$$

6.3 Speed of sound

The Newton-Laplace's equation, which gives the speed of sound in an ideal gas, is:

$$
c = \sqrt{\frac{K}{\rho}} \tag{65}
$$

$$
= \sqrt{\frac{\gamma P}{\rho}} \tag{66}
$$

where K is known as the isentropic[6] bulk modulus or the coefficient of stiffness.

Consider the ideal gas law.

$$
PV = nRT \tag{67}
$$

$$
P = \frac{nM}{V}\frac{RT}{M} \tag{68}
$$

$$
P = \rho\frac{R}{M}T = \rho R_{\text{specific}}T \tag{69}
$$

$$
\frac{P}{\rho} = \frac{R}{M}T \tag{70}
$$

Sub into the expression for c^2.

$$
c^2 = \frac{\gamma\overline{P}}{\overline{\rho}} \tag{71}
$$

$$
= \gamma\frac{R}{M}T \tag{72}
$$

[6](Ideal) sound wave propagation is an adiabatic and reversible process, but NOT isothermal since temperature can fluctuate rapidly during propagation (air expansion and compression).

Check that this is the equation for speed of sound of an ideal gas: $c = \sqrt{\gamma \frac{R}{M} T}$. It is only dependent on temperature, and gas composition. From [6]:
In general, the more rigid (or less compressible) the medium, the faster the speed of sound. This observation is analogous to the fact that the frequency of simple harmonic motion is directly proportional to the stiffness of the oscillating object as measured by k, the spring constant. The greater the density of a medium, the slower the speed of sound. This observation is analogous to the fact that the frequency of a simple harmonic motion is inversely proportional to m, the mass of the oscillating object. The speed of sound in air is low, because air is easily compressible. Because liquids and solids are relatively rigid and very difficult to compress, the speed of sound in such media is generally greater than in gases.

7 Arriving at the wave equation: Part 2

Previously, we considered a zero mean flow ($\overline{u}$=0). Now, we consider a non-zero mean flow. From [3]:

8 Pressure wave equation

As shown earlier, the homogeneous wave equation is written as:

$$\nabla^2 f(\mathbf{x}, t) = \frac{1}{c^2} \frac{\partial^2 f(\mathbf{x}, t)}{\partial t^2} \tag{73}$$

The wave equation is the simplest example of a hyperbolic differential equation. The wave equation alone does not specify a physical solution; a unique solution is usually obtained by setting a problem with further conditions, such as initial conditions, which prescribe the amplitude and phase of the wave. or boundary conditions for problems occurring in enclosed space. Boundary value solutions represent standing waves, or harmonics, analogous to the harmonics of musical instruments.

The basic wave equation is a linear differential equation and so it will adhere to the superposition principle. This means that the net displacement caused by two or more waves is the sum of the displacements which would have been caused by each wave individually. In addition, the behavior of a wave can be analysed by breaking up the wave into components, e.g. the Fourier transform breaks up a wave into sinusoidal components.

8.1 Arriving at the Helmholtz equation / eigenvalue problem

If one assumes that space-time can be decoupled, and applies the technique of separation of variables: $f(\mathbf{x}, t) = P(\mathbf{x})T(t)$. Then, we have:

$$\begin{aligned}
\nabla^2 (P(\mathbf{x})T(t)) &= \frac{1}{c^2} \frac{\partial^2 (P(\mathbf{x})T(t))}{\partial t^2} \\
T(t)\nabla^2 P(\mathbf{x}) &= \frac{P(\mathbf{x})}{c^2} \frac{\partial^2 T(t)}{\partial t^2} \\
\underbrace{\frac{\nabla^2 P(\mathbf{x})}{P(\mathbf{x})}}_{\text{spatial}} &= \underbrace{\frac{1}{c^2} \frac{1}{T(t)} \frac{\partial^2 T(t)}{\partial t^2}}_{\text{temporal}} = -k^2
\end{aligned} \tag{74}$$

where k is a constant. This has to be true, since the left-hand side of Eqn. 74 is dependent only on $\mathbf{x}$, while the right-hand side is dependent only on t. Therefore, the only possibility for them to be equal to each other at any value of $\mathbf{x}$ and t, is to be equal to a constant, say $-k^2$ [7]. See Section 8.2 and note how the temporal solution gives rise to the eigenvalue $-\frac{w^2}{c^2}$, which we can equate to $-k^2$.

8.2 Solutions to the wave equation (temporal)

The temporal component usually takes the form $T(t) = e^{-i\omega t}$. This is especially so if the solution oscillates in time with a well-defined constant angular frequency, ω. In which case, ω is the eigenfrequency, and its associated solution is the eigenmode. See Section 8.5.

Sub $T(t) = e^{-i\omega t}$ into Eqn. 74:

$$\begin{aligned}
\frac{\nabla^2 P}{P} &= \frac{1}{c^2}\frac{1}{e^{-i\omega t}}\frac{\partial^2(e^{-i\omega t})}{\partial t^2} \\
&= \frac{1}{c^2}\frac{1}{e^{-i\omega t}} i^2\omega^2 e^{-i\omega t} \\
\nabla^2 P &= -\frac{\omega^2}{c^2} P \\
\nabla^2 P &= -k^2 P
\end{aligned} \tag{75}$$

$k = \frac{\omega}{c}$ is the wavenumber, i.e. the number of full waves that exist at a time instant in unit length. Notice that we arrive at Eqn. 75, which is the same as Eqn. 74.

8.3 Solutions to the wave equation (spatial: 1-D)

From Eqn. 74, the time-independent wave equation is:

$$\nabla^2 P(\mathbf{x}) = -k^2 P(\mathbf{x}) \tag{76}$$

Notice that this is also an eigenvalue problem for the laplace operator, otherise known as the Helmholtz equation. k is also known as the eigenvalue (aside from being a wavenumber), and P is the eigenfuncion.

And if the right-hand side of Eqn. 76 is equal to 0, we have a Laplace equation: $\nabla^2 P(\mathbf{x}) = 0$.[8]

Now, one can simplify the equation further by considering only a single dimension, x. Eqn. 76 is written as:

$$\nabla^2 P(x) = -k^2 P(x)$$

Solutions have to be of the form whereby upon being differentiated twice, have the same form as the original solution term(s). It is thus obvious that the solutions can be either sinusoidal or exponential (one can be expressed in terms of another by the Euler's formula, $e^{i\theta}$ anyway). A logical choice would then be:

$$\begin{aligned}
P(x) &\propto e^{\pm ikx} \\
&= Ae^{ikx} + Be^{-ikx}
\end{aligned}$$

[7] $-k^2$ is chosen to be this constant because if we take the square root, we have $\sqrt{-k^2} = ik$. Makes analysis easier. Nothing stops you from using A or anything to represent the constant though!

[8] Note that Poisson ($\nabla^2 u = f$) and Helmholtz ($\nabla^2 u = -ku$) are both second-order PDE. However, for Poisson's equation, function f need not be related to function u. But I would not say that Helmholtz is a subset of Poisson either?

8.4 Solutions to the wave equation (spatial: 2-D polar)

Let us consider a two-dimensional polar coordinates, since it is more relevant to acoustic studies in an annular configuration. In experiments, we would place dynamic pressure sensors in the combustion chamber at different azimuthal positions, but identical axial and radial positions [7].

Let us apply separation of variables again: $P(\mathbf{x}) = R(r)\Theta(\theta)$, with $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.

Eqn. 76: $\nabla^2 P(x) = -k^2 P(x)$ becomes:

$$\begin{aligned}
\Theta\frac{\partial^2 R}{\partial r^2} + \frac{\Theta}{r}\frac{\partial R}{\partial r} + \frac{R}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2} &= -k^2 R\Theta \\
\frac{R''}{R} + \frac{R'}{Rr} + \frac{\Theta''}{\Theta r^2} &= -k^2 \\
\frac{R''}{R}r^2 + \frac{R'}{R}r + \frac{\Theta''}{\Theta} &= -k^2 r^2 \\
\underbrace{\frac{R''}{R}r^2 + \frac{R'}{R}r + k^2 r^2}_{\text{radial}} &= \underbrace{-\frac{\Theta''}{\Theta}}_{\text{azimuthal}} = n^2 \qquad (77) \\
& \qquad (78)
\end{aligned}$$

Again, notice that the left-hand side of Eqn. 77 is dependent only on r, while the right-hand side is dependent only on θ. Therefore, the only possibility for them to be equal to each other at any value of r and θ, is to be equal to a constant, say n^2.

The azimuthal component can be solved easily with the boundary condition $\Theta(\theta) = \Theta(\theta + 2\pi)$, since in an annular configuration, periodicity must be adhered! n has to be an integer.

Notice that the radial component is similar to the Bessel equation.

$$\begin{aligned}
\frac{R''}{R}r^2 + \frac{R'}{R}r + k^2 r^2 - n^2 &= 0 \\
r^2 R'' + rR' + (k^2 r^2 - n^2)R &= 0 \qquad (79)
\end{aligned}$$

Compare this with the Bessel equation:

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0 \qquad (80)$$

Consider a change in variable: $\rho = kr$, and re-express $\frac{\partial R}{\partial r}$ and $\frac{\partial^2 R}{\partial r^2}$:

$$\begin{aligned}
\frac{\partial R}{\partial r} &= \frac{\partial R}{\partial \rho}\frac{\partial \rho}{\partial r} \\
&= \frac{\partial R}{\partial \rho} k \\
\frac{\partial^2 R}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial R}{\partial r}\right) \\
&= \frac{\partial\left(\frac{\partial R}{\partial r}\right)}{\partial \rho}\frac{\partial \rho}{\partial r} \\
&= \frac{\partial\left(\frac{\partial R}{\partial \rho}k\right)}{\partial \rho} k \\
&= \frac{\partial^2 R}{\partial \rho^2} k^2
\end{aligned}$$

Then sub into Eqn. 79.

$$\begin{aligned}
\left(\frac{\rho}{k}\right)^2 \frac{\partial^2 R}{\partial \rho^2} k^2 + \left(\frac{\rho}{k}\right)\frac{\partial R}{\partial \rho} k + (\rho^2 - n^2)R = 0 \\
\rho^2 \frac{\partial^2 R}{\partial \rho^2} + \rho \frac{\partial R}{\partial \rho} + (\rho^2 - n^2)R = 0
\end{aligned} \tag{81}$$

Compare Eqn. 81 with Eqn. 80, and notice that we have the same form. And we know that the general solution to the Bessel equation Eqn. 80 is of the form [3]:

$$R(r) = c_1 J_n(\rho) + c_2 Y_n(\rho) \tag{82}$$

where $J_n(\rho)$ and $Y_n(\rho)$ are the Bessel functions of the first and second kind respectively. Since Y_n is singular at $r = 0$, c_2 has to be zero so that we do not end up with $R\,|_{r=0}$ = undefined!
Also, remember that n has to be an integer. This corroborates well with $\alpha = n$ in Eqn. 80 being an integer, since we are solving the Laplace equation in cylindrical coordinates. Unsurprisingly, Bessel functions for integer α are also known as cylinder functions or cylindrical harmonics. See Section 14.2 for more on Bessel functions.

8.5 General solution of plane waves: Eigenmodes / Fourier series / Galerkin approach

This is a collection of the Galerkin approaches adopted in various papers [2], [3], [7]–[9].

From [3]:

$$p'(x,t) = \sum_{m=1}^{\infty} \eta_m(t)\psi_m(x)$$

Expanding the pressure perturbation as a Galerkin series where the functions $\psi_m(x)$ are the eigensolutions or normal modes of a homogeneous wave equation that satisfy the same boundary conditions as p'. In general, $\psi_m(x)$ are orthogonal to each other.

From [8]:
The Galerkin technique makes use of the fact that any function in a domain can be expressed as a superposition of expansion functions which form a complete basis in that domain. The basis functions are

chosen such that they satisfy the boundary conditions. However, the choice of the basis functions is not unique.

From [9]:
Making use of the Galerkin technique, the pressure field can be expanded in a series of orthogonal basis functions:

$$p(\theta, t) = \sum_{n=1}^{\infty} \eta_n(t)\psi_n(\theta)$$

One considers the situation where the thermoacoustic coupling operating in the combustion chamber results from the combination of the eigenmodes sharing the n_{th} eigenvalue, which has the highest linear growth rate. Assuming that this growth rate (the $\Re$ part of the n_{th} eigenvalue) is small compared to the oscillation frequency (the $\Im$ part of the eigenvalue), it is possible to approximate the pressure field by:

$$p(\theta, t) = \eta_1(t)\cos(n\theta) + \eta_2(t)\sin(n\theta)$$

From [7]:
Considering the periodic nature of annular geometries, it is natural to decompose the acoustic pressure field p as a Fourier series:

$$p(r, \theta, z, t) = \sum_{m=1}^{\infty} \mu_{m,a}(t) f_m(r, z)\cos(m\theta) + \mu_{m,b}(t) f_m(r, z)\sin(m\theta) \tag{83}$$

$$p(\theta, t) = \sum_{m=1}^{\infty} \eta_{m,a}(t)\cos(m\theta) + \eta_{m,b}(t)\sin(m\theta) \tag{84}$$

Eqn. 83 considers dependence on the radial r, and axial z directions. If all measurements and analysis are done for one fixed value of r and z, one may consider Eqn. 84, where $\eta_{m,a}(t) = \mu_{m,a}(t)f_m(r, z)$. Likewise for $\eta_{m,b}(t)$.

The temporal Fourier transforms of $\eta_{m,a}(t)$ and $\eta_{m,b}(t)$, $\mathcal{F}(\eta_{m,a}(t))$ and $\mathcal{F}(\eta_{m,b}(t))$ generally show multiple distinct maxima. The frequencies at which these maxima occur are associated with the acoustic modes of the annular combustion chamber. The flames which are distributed along the circumference can interact constructively with the combustion chamber's acoustics. In this case, the system can be brought onto a limit cycle. This is usually characterised by one pair of dominant azimuthal Fourier coefficients, $\eta_{n,a}(t)$ and $\eta_{n,b}(t)$ with a dominant limit cycle frequency ω_n. This type of behaviour is typical of weakly damped/amplified oscillators which feature two different time scales: the oscillation frequency $T_n = \frac{2\pi}{\omega_n}$, and the relaxation time which is associated to the dynamics of the amplitude variations and which is much longer than T_n. Therefore, the space-time dependence of the acoustic field can be reasonably approximated by:

$$p(\theta, t) \approx \eta_a(t)\cos(n\theta) + \eta_b(t)\sin(n\theta)$$

From [2]:
In a duct $\mathfrak{D}$ of constant cross section $\mathfrak{A}$, the reduced wave (or Helmoltz) equation may be solved by means of a series expansion in a particular family of solutions, called modes. They are related to the eigensolutions of the two-dimensional Laplace operator acting on a cross section. Modes are interesting mathematically because they form, in general, a complete basis by which any solution can be represented. Physically, modes are interesting because they are solutions in their own right, not just mathematical building blocks, and by their simple structure the usually complicated behaviour of the total field is more easily understood.

Consider the acoustic field:

$$p(\mathbf{x},t) = p(\mathbf{x},\omega)e^{i\omega t}, \qquad \mathbf{u}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},\omega)e^{i\omega t}$$

which satisfies in the duct ($\mathbf{x} \in \mathfrak{D}$),

$$\begin{aligned} \nabla^2 p + \omega^2 p &= 0 \\ i\omega\mathbf{u} + \nabla p &= 0 \end{aligned}$$

Self-similar solutions (called modes) of the form $p(x,y,z) = \phi(x)\psi(y,z)$ exist for $\phi(x) = e^{-ikx}$ with particular values of k and associated functions ψ. This leads to general solutions given by:

$$p(x,y,z) = \sum_{n=0}^{\infty} C_n \psi_n(y,z) e^{-ik_n x}$$

where ψ_n are the eigenfunctions of the Laplace operator reduced to the area, $\mathfrak{A}$, i.e. solutions of the two-dimensional problem

$$-\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi = \alpha^2\psi \text{ for } (y,z) \in \mathfrak{A}$$

where α^2 is the corresponding eigenvalue.
The axial wave number k is given by one of the square roots $k = \pm\sqrt{\omega^2 - \alpha^2}$ (+ for right running, and - for left running). Each term in the series expansion, i.e. $\psi_n(y,z)e^{-ik_n x}$ is called a duct mode. If the duct cross section is circular or rectangular and the boundary condition is uniform everywhere, the solutions of the eigenvalue problem are relatively simple and may be found by separation of variables. These eigensolutions consist of combinations of exponentials and Bessel functions in the circular case or combinations of trigonometric functions in the rectangular case.

8.6 General solution of plane waves: algebraic

Recall that a one-dimensional, simplified wave equation is:

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2} \tag{85}$$

where P is pressure wave, or any other quantity, e.g. displacement...

Solving by d'Alembert (Yes, he is French!), we consider the following change in variables:

$$\xi = x - ct \qquad \eta = x + ct$$

So to express p: $p(x,t) \to p(\xi,\eta)$. We know that p is 2nd order with respect to x amd t??? if 1st order, then is trivial solution (=0) ???
First, re-express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} \\ &= -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta} \end{aligned}$$

which we may proceed to compute $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial t^2}$

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\right)\\ &= \left(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\right)\\ &= \frac{\partial^2}{\partial \xi^2}+2\frac{\partial^2}{\partial \xi \partial \eta}+\frac{\partial^2}{\partial \eta^2} \qquad (86)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial t^2} &= \frac{\partial}{\partial t}\left(-c\frac{\partial}{\partial \xi}+c\frac{\partial}{\partial \eta}\right)\\ &= \left(-c\frac{\partial}{\partial \xi}+c\frac{\partial}{\partial \eta}\right)\left(-c\frac{\partial}{\partial \xi}+c\frac{\partial}{\partial \eta}\right)\\ &= c^2\frac{\partial^2}{\partial \xi^2}-2c^2\frac{\partial^2}{\partial \xi \partial \eta}+c^2\frac{\partial^2}{\partial \eta^2} \qquad (87)\end{aligned}$$

Then, sub Eqn. 86 and Eqn. 87 into Eqn. 85.

$$\left(c^2\frac{\partial^2}{\partial \xi^2}-2c^2\frac{\partial^2}{\partial \xi \partial \eta}+c^2\frac{\partial^2}{\partial \eta^2}\right)p = c^2\left(\frac{\partial^2}{\partial \xi^2}+2\frac{\partial^2}{\partial \xi \partial \eta}+\frac{\partial^2}{\partial \eta^2}\right)p \qquad (88)$$

$$\frac{\partial^2 p}{\partial \xi \partial \eta} = 0 \qquad (89)$$

We arrive at the conclusion that $p(\eta, \xi)$ has first order dependence with respect to both η and ξ. The general solution is thus of the form:

$$\begin{aligned}p(x,t) &= f(\eta)+g(\xi)\\ &= f(x-ct)+g(x+ct)\end{aligned}$$

where f and g are arbitrary functions, with f representing a right-travelling wave, and g representing a left-travelling wave.
The forms of the f and g functions are determined by initial conditions, and boundary conditions.
Since pressure propagation are essentially travelling waves, a logical choice (in most cases) would be sinusoidal forms: $f = \cos(kx-\omega t)$ and $g = \cos(kx+\omega t)$ [9].

9 Annular Multi-Microphone Method

Let us consider the case where the quantity of interest in the wave equation is pressure fluctuations. p_{θ_i} is a pressure measurement data taken at position θ_i, so it has to be $\Re$ with no $\Im$ parts. (If dynamic pressure measurements are taken, strictly speaking, we should write it as p'_{θ_i}.)
So if we are to consider expressing it with an imaginary number, say z, it will be:

$$\begin{aligned}p_{\theta_i} &= \Re(z)\\ &= \frac{1}{2}(z+z^*)^{10}\end{aligned}$$

Let $z = Fe^{in\theta}$ where F is the complex amplitude given by: $F = F_r + iF_i$. The i and r subscripts denote real and imaginary.

[9] you can think of it loosely as: $\sin(x-ct) \implies \sin(k(x-ct)) \implies \sin(kx - \frac{w}{c}ct) \implies \sin(kx-\omega t)$.
[10] Recall that: $z + z^* = (x+iy)+(x-iy) = 2\Re = 2x$

Then, $z = (F_r + iF_i)e^{in\theta} = (F_r + iF_i)(\cos n\theta + i\sin n\theta) = (F_r\cos n\theta - F_i \sin n\theta) + i(F_r \sin n\theta + F_i \cos n\theta)$.
Then, its complex conjugate, $z^* = (F_r \cos n\theta - F_i \sin n\theta) - i(F_r \sin n\theta + F_i \cos n\theta)$.
And,

$$\begin{aligned}
p_{\theta_i} &= \frac{1}{2}(z + z^*) \\
&= \frac{1}{2}(2F_r \cos n\theta - 2F_i \sin n\theta) \\
&= F_r \cos n\theta - F_i \sin n\theta
\end{aligned}$$

which of course, we can also write as:

$$p_{\theta_i} = \frac{1}{2}(2Fe^{in\theta})$$

and then since P_{θ_i} has to be real, we discard the imaginary part. Does that mean $F_r \sin\theta + F_i \cos\theta$ is zero?

$$\begin{aligned}
p_{\theta_i} &= Ae^{-in\theta_k} + Be^{in\theta_k} \qquad \text{(combination of one CW and one CCW wave)} \\
&= A[\cos(n\theta_k) - i\sin(n\theta_k)] + B[\cos(n\theta_k) + i\sin(n\theta_k)] \\
&= (A + B)\cos(n\theta_k) - i(A - B)\sin(n\theta_k) \\
&= F_r \cos(n\theta_k) - iF_i \sin(n\theta_k)
\end{aligned}$$

In matrix form, we have:

$$p_{\theta_i} = \begin{bmatrix}\cos(n\theta_i) & -\sin(n\theta_i)\end{bmatrix}\begin{bmatrix}F_r \\ F_i\end{bmatrix}$$

That's decomposition of p_i into two modes? Or rather, one should think of it as projection. The first basis vector is $\cos(n\theta_i)$ and the second basis vector is $\sin(n\theta_i)$. If you consider that $\cos(n\theta_i)$ is orthonormal to $\sin(n\theta_i)$, then they are orthonormal basis. Note also that sinusoidal waves of different frequencies are orthogonal to each other.

Then p at a time instant, t, is:

$$p(t) = \sum_{k=1}^{M} F_r(t)\cos(n\theta_k) + iF_i(t)\sin(n\theta_k)$$

If we have microphones at more than 2 azimuthal positions, we have an over-constrained problem (2 unknowns, F_r and F_i, and M number of equations (for M number of microphones) at each time instant. Other known variables: pressure $p(t)$, azimuthal wavenumber, n, and microphone azimuthal positions, θ_i . Also, note that microphones need not be equidistant! May be spread out unevenly around the circumference.

Recall from [7], we have:

$$p(\theta, t) \approx \eta_a(t)\cos(n\theta) + \eta_b(t)\sin(n\theta)$$

Similar in form!

How to determine the azimuthal wave number, n? By residual of least means squared (you have to calculate that at each time insant t!), to see what value of n fits better. Or, estimate from the geometry. E.g. $\omega = nc(T)/R$, where c(T) is the sound speed at temperature T, R is the mean radius of the combustor, and n is the harmonics. n=1 is one wavelength in a circumference, n=2 is two wavelengths in a circumference. It really depends which is the dominant azimuthal wave number. It may be when $n = 1$ or when $n = 2$. Or may be a sum of $n = 2$ and $n = 3$.

10 Annular Multi-Microphone Method: Numerical Implementation

10.1 Code input

- pressure matrix (rows = time, columns = angles), $P = \begin{bmatrix} P_{\theta_1,t_1} & \cdots & P_{\theta_k,t_1} & \cdots & P_{\theta_M,t_1} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ P_{\theta_1,t_j} & \cdots & P_{\theta_k,t_j} & \cdots & P_{\theta_M,t_j} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ P_{\theta_1,t_N} & \cdots & P_{\theta_k,t_N} & \cdots & P_{\theta_M,t_N} \end{bmatrix}$
- azimuthal position of the pressure measurements (column vector), $\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_k \\ \vdots \\ \theta_M \end{bmatrix}$
- azimuthal wavenumber, n

10.2 Code output

- complex wave amplitude (column vector), $F = \begin{bmatrix} F_{r,t_1} + iF_{i,t_1} \\ \vdots \\ F_{r,t_j} + iF_{i,t_j} \\ \vdots \\ F_{r,t_N} + iF_{i,t_N} \end{bmatrix}$
- complex wave amplitude at time t_j (column vector), $x = \begin{bmatrix} F_{r,t_j} \\ F_{i,t_j} \end{bmatrix}$

10.3 Description

Aim is to solve F at each time instant t_j via the equation $P(j,:)^T = Ax$ where $A = \begin{bmatrix} \cos n\theta_1 & -\sin n\theta_1 \\ \vdots & \vdots \\ \cos n\theta_k & -\sin n\theta_k \\ \vdots & \vdots \\ \cos n\theta_M & -\sin n\theta_M \end{bmatrix}$

E.g. we have 3 microphone positions, $\theta_1, \theta_2, \theta_3$. Then, at any time instant, t_j:

$$\begin{aligned} P(j,:)^T &= Ax \\ \begin{bmatrix} P_{\theta_1,t_j} \\ P_{\theta_2,t_j} \\ P_{\theta_3,t_j} \end{bmatrix} &= \begin{bmatrix} \cos n\theta_1 & -\sin n\theta_1 \\ \cos n\theta_2 & -\sin n\theta_2 \\ \cos n\theta_3 & -\sin n\theta_3 \end{bmatrix} \begin{bmatrix} F_{r,t_j} \\ F_{i,t_j} \end{bmatrix} \end{aligned} \tag{90}$$

To solve equation of the form $b = Ax$, as in Eqn. 90, the least squares method using QR decomposition is used. Perform a QR decomposition on A. Then, we have $[q, r] = qr(A)$ where q is the orthogonal matrix[11], and r is the upper triangular matrix.

For $j = 1 : N$ (looping through time t_j: t_1 to t_N), let $b = P(j,:)^T$

$$\begin{aligned} b &= Ax \\ b &= qrx \\ q^T b &= rx \\ x &= r \backslash q^T b \end{aligned}$$

Then,

$$\begin{aligned} F(j) &= x(1) + ix(2) \\ &= F_{r,t_j} + iF_{i,t_j} \end{aligned}$$

11 Longitudinal Multi-Microphone Method (MMM)

Along an axial (longitudinal) direction, we may express the pressure distribution as the sum of a forward propagating and backward propagating wave. We shall solve the wave amplitudess in the frequency domain (the rationale of doing so shall be explained in section 13).

Let us consider the pressure, p, at a particular axial position, x, and a time instant, t, to be written as:

$$\begin{aligned} p(t,x) &= \frac{1}{2}\hat{p}(x)e^{i\omega t} + \text{cte} \\ &= \frac{1}{2}\underbrace{\left(Fe^{ik_p x} + Ge^{ik_m x}\right)}_{\hat{p}(x)} e^{i\omega t} + \text{cte} \end{aligned}$$

where F and G are complex wave amplitudes,
with the related wavenumbers $k_p = \frac{k}{1+M}$, $k_p = \frac{k}{1-M}$, $k = \frac{\omega}{c}$,
and where c is speed of sound, and M is Mach number.

One could obviously write it as $Fe^{-ik_p x} + Ge^{ik_m x}$ to clearly demonstrate that the first term corresponds to a forward propagating wave, but it may make further manipulation less tidy. Another thing to note is that by convention, the forward propagating wave has a negative wavenumber, i.e. it is usually written as e^{-ikx}. Lastly, note that there is already a separation of variables: spatial and temporal. This spatial term is written in the Fourier space / frequency domain, so it is time independent!

For N number of microphones, we may then write it as:

$$\begin{bmatrix} \hat{p}(x_1) \\ \vdots \\ \hat{p}(x_j) \\ \vdots \\ \hat{p}(x_N) \end{bmatrix} = \begin{bmatrix} e^{ik_p x_1} & e^{ik_m x_1} \\ \vdots & \vdots \\ e^{ik_p x_j} & e^{ik_m x_j} \\ \vdots & \vdots \\ e^{ik_p x_N} & e^{ik_m x_N} \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix}$$

The question is then: how to get $\hat{p}$, and ω? From spectrum analysis! $\hat{p} = \sum$

What about the speed of sound, c? by finding the adiabatic flame temperature, and then $c = \sqrt{\gamma R T_{adiabatic}}$

[11]Recall that: $q^T q = qq^T = \mathbb{I}$

12 Longitudinal Multi-Microphone Method (MMM): Numerical Implementation

12.1 Code input

- pressure matrix (rows = time, columns = axial position), $P = \begin{bmatrix} P_{x_1,t_1} & \cdots & P_{x_k,t_1} & \cdots & P_{x_M,t_1} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ P_{x_1,t_j} & \cdots & P_{x_k,t_j} & \cdots & P_{x_M,t_j} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ P_{x_1,t_N} & \cdots & P_{x_k,t_N} & \cdots & P_{x_M,t_N} \end{bmatrix}$
- axial position of the pressure measurements (column vector), $x\_mic = \begin{bmatrix} x\_mic_1 \\ \vdots \\ x\_mic_k \\ \vdots \\ x\_mic_M \end{bmatrix}$
- number of microphones, N
- speed of sound, c
- Mach number, M

12.2 Code output

- complex wave amplitude of forward propagating wave, F
- complex wave amplitude of backward propagating wave, G
- (dominant, time-averaged) angular frequency, ω
- (in the frequency domain) complex amplitude of pressure fluctuations at frequency ω, $\hat{P}$

12.3 Description

Aim is to solve F and G in the frequency domain, via the equation $\hat{P} = Ax$ where $A = \begin{bmatrix} e^{ik_p x_1} & e^{ik_m x_1} \\ \vdots & \vdots \\ e^{ik_p x_j} & e^{ik_m x_j} \\ \vdots & \vdots \\ e^{ik_p x_N} & e^{ik_p x_N} \end{bmatrix}$ and $x = \begin{bmatrix} F \\ G \end{bmatrix}$. Solving this is very straightforward, via the least squares method using QR decomposition.

Let $b = \hat{P}$

$$
\begin{aligned}
b &= Ax \\
b &= qrx \\
q^T b &= rx \\
x &= r \backslash q^T b
\end{aligned}
$$

Then,

$$F = x(1)$$
$$G = x(2)$$

13 Multi-Microphone Method: comparison of different approaches

Note the different approaches of the multi-microphone method (MMM) in an annular and longitudinal configuration. In an annular, you are solving in the time domain. In a longitudinal, you are solving it in the frequency domain. (By doing so, you are assuming that the integrals or summations are Lebesgue summable, which is a reasonable assumption for real-life signals!)
But Why the different approaches?

Case of annular
In an annular configuration, the modes may rotate round the circumference (spinning mode). What it means is that at each time instant, the (pressure) wave is propagating, with the nodal and anti-nodes shifting along the circumference, so trying to solve for F and G in the frequency domain will not work! Keep in mind that when you are in frequency domain, you are freezing time! (Recall that one of the conditions of using Fourier transform is that we are assuming that the (sample) function is representative of the actual function, i.e. the (sample) function is a period of the actual function. To refer to notes on Integral Transforms.)
So, by solving F and G in the frequency domain, what you get is the response (pressure amplitudes) at each frequency! but we know that the modes may be spinning (i.e. shifting along the circumference at different time instants, so time dependent). So it only makes sense to solve F and G in the real-time domain, i.e. $F = F(t)$, $G = G(t)$... so that's why we end up solving a matrix of the wave amplitudes: $F = \begin{bmatrix} F(t_1) \\ \vdots \\ F(t_j) \\ \vdots \\ F(t_N) \end{bmatrix}$. Likewise for G.
So, in short: in an annular configuration, we want to reconstruct the mode in the chamber, but the waves can rotate, i.e., move in space in time passes (so time dependent). So solution is logically, transient (we solve at each time instant)!

Case of longitudinal
For longitudinal mode, we solve the Riemann invariants in the frequency domain. The idea is that we will have invariants (time independent), so that's neater! Especially when we are interested in the frequency response. Since in most cases, what we are interested is at a given frequency, what the response of the system (pressure amplitudes) is.

14 Appendix

14.1 Orthonormal

To show that it is orthonormal, consider the inner product of two functions, $\phi(x)$ and $\psi(x)$: $\langle \phi(x), \psi(x) \rangle$. If orthonormal, we need to have:

$$\begin{gathered}
\langle \phi(x), \psi(x) \rangle = 0 \\
\text{and} \\
\|\phi(x)\|_{L_2} = \|\psi(x)\|_{L_2} = 1 \\
\text{OR write it as:} \\
\int_a^b \phi(x)\psi(x)\,dx = 0 \\
\text{and} \\
\left[\int_a^b \mid \phi(x) \mid^2 \; dx\right]^{\frac{1}{2}} = \left[\int_a^b \mid \psi(x) \mid^2 \; dx\right]^{\frac{1}{2}} = 1
\end{gathered}$$

In the case of $\cos\theta$ and $\sin\theta$, the inner product is:

$$\begin{aligned}
\int_0^{2\pi} \cos\theta \sin\theta \, d\theta &= \left[-\cos^2\theta\right]_0^{2\pi} - \int_0^{2\pi} \cos\theta \sin\theta \, d\theta \qquad (i.p.p.) \\
2\int_0^{2\pi} \cos\theta \sin\theta \, d\theta &= [-1 - (-1)] \\
\int_0^{2\pi} \cos\theta \sin\theta \, d\theta &= 0
\end{aligned}$$

which is unsurpising, since $\cos\theta$ is an even function, and $\sin\theta$ is an odd function. A product of these two functions gives an even function. Integration over a period would of course be zero.

While $\|\cos\theta\|_{L_2}$ and $\|\sin\theta\|_{L_2}$:

$$\begin{aligned}
\left[\int_0^{2\pi} \cos^2\theta \, d\theta\right]^{\frac{1}{2}} &= \left[\int_0^{2\pi} \left(\frac{\cos 2\theta + 1}{2}\right) d\theta\right]^{\frac{1}{2}} \\
&= \sqrt{\left[\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right]_0^{2\pi}} \\
&= \sqrt{\pi} \\
\left[\int_0^{2\pi} \sin^2\theta \, d\theta\right]^{\frac{1}{2}} &= \left[\int_0^{2\pi} \left(\frac{1 - \cos 2\theta}{2}\right) d\theta\right]^{\frac{1}{2}} \\
&= \sqrt{\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}} \\
&= \sqrt{\pi}
\end{aligned}$$

Fourier series is essentially just a series of expressing a periodic function in terms of sinusoidal basis functions...

14.2 Bessel Equations

14.2.1 Overview

As in Eqn. 80, Bessel's differential equation has the form:

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

and Bessel functions are canonical solutions $y(x)$ of the Bessel's differential equation. $\alpha \in \mathbb{C}$ is the order of the Bessel function. But usually, we are only interested in $\alpha \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$.

Bessel functions for integer $\alpha \in \mathbb{Z}$ are also known as cylinder functions or cylindrical harmonics because they appear in the solution to Laplace's equation $\nabla^2_{\mathbf{r}} f = -kf$ in cylindrical coordinates.

Bessel functions for half-integer $\alpha \in \mathbb{Z}$ are known as spherical Bessel functions because they appear in the solution to the Helmholtz equation in spherical coordinates.

A quick recap, in case you are wondering how we get mired in a second-order PDE.

- started off from the wave equation as in Eqn. 73:

$$\nabla^2 f(\mathbf{x}, t) = \frac{1}{c^2}\frac{\partial^2 f(\mathbf{x}, t)}{\partial t^2}$$

- Next, a spatial-temporal separation $f(\mathbf{x}, t) = P(\mathbf{x})T(t)$ as in Eqn. 74

$$\underbrace{\frac{\nabla^2 P(\mathbf{x})}{P(\mathbf{x})}}_{\text{spatial}} = \underbrace{\frac{1}{c^2}\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2}}_{\text{temporal}} = -k^2$$

- Then, a radial-azimuthal separation $P(\mathbf{x}) = R(r)\Theta(\theta)$ as in Eqn. 77

$$\underbrace{\frac{R''}{R}r^2 + \frac{R'}{R}r + k^2r^2}_{\text{radial}} = \underbrace{-\frac{\Theta''}{\Theta}}_{\text{azimuthal}} = n^2$$

- Then, a change in variables $\rho = kr$ to arrive at the Besel equation as in Eqn. 81

$$\rho^2\frac{\partial^2 R}{\partial \rho^2} + \rho\frac{\partial R}{\partial \rho} + (\rho^2 - n^2)R = 0$$

14.2.2 Solutions to Bessel equations: Bessel functions

Since a Bessel equation is a seoncd-order PDE, we need to have two linearly independent solutions. The general solution is:

$$R(r) = c_1 J_n(rk) + c_2 Y_n(rk)$$

where J_n and Y_n are Bessel functions of the first kind of order n (or α), and second kind of order n (or α), respectively. They are the two linearly independent solutions.

Case: $n \in \mathbb{Z}$ (cylindrical functions)
Bessel function of the first kind with $n \in \mathbb{Z}$, is as shown in Fig. 1. Notice the following:

- $J_0\,|_{r=0} = 1$
- For any order n, J_n crosses the x-axis infinitely times. Thus, there are infinitely many possible solutions to the Bessel equation as well.

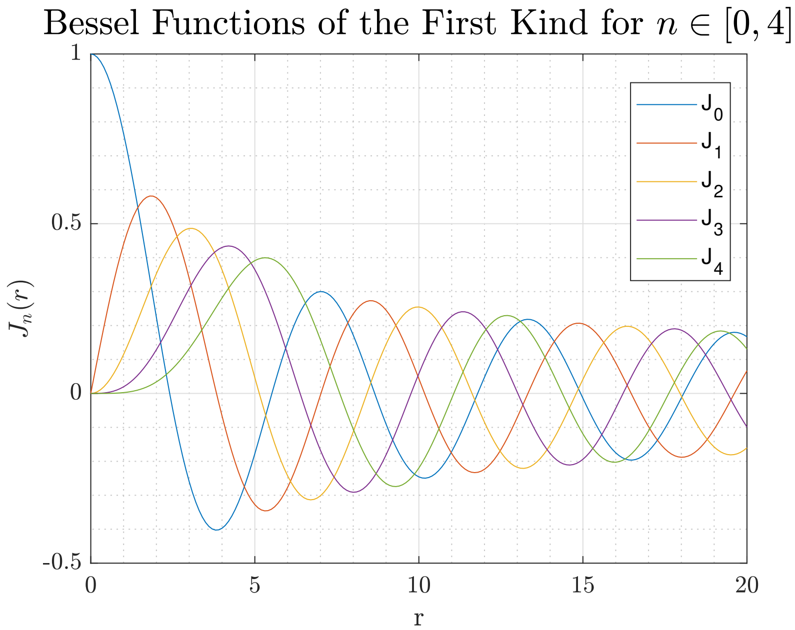

Fig. 1: Plots of J_n.

Bessel function of the second kind with $n \in \mathbb{Z}$ is as shown in Fig. 2. Notice the following:

- For any order n, $Y_n\,|_{r=0}$ = undefined.
- For any order n, Y_n crosses the x-axis infinitely times. Thus, there are infinitely many possible solutions to the Bessel equation as well (similar to J_n).

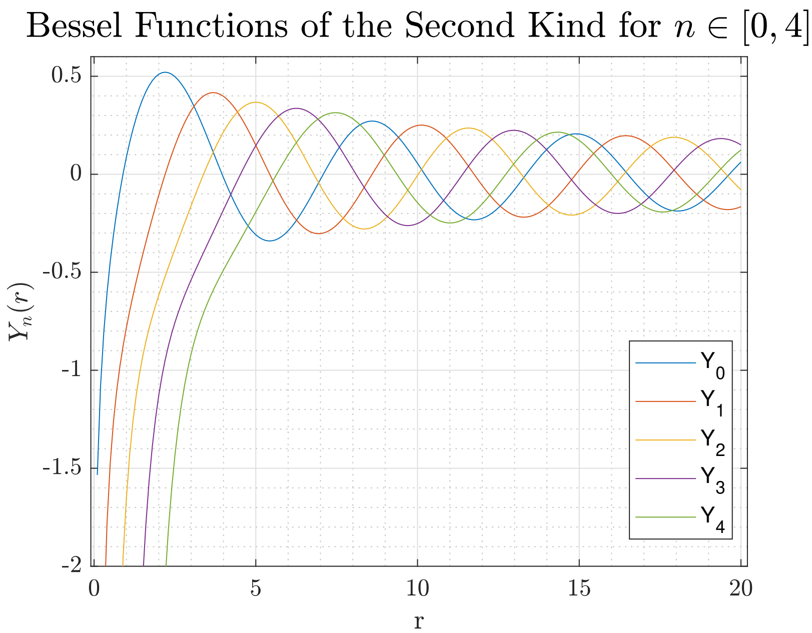

Fig. 2: Plots of Y_n.

Referring to Section 8.4, we can see why Y_n is not part of the solution to the wave equation, since we have $Y_n\,|_{r=0}$ = undefined.

Case: $n \in \mathbb{Z} + \frac{1}{2}$ (spherical functions)

15 References

[1] T. C. Lieuwen, *Unsteady Combustor Physics*. Cambridge: Cambridge University Press, 2012, ISBN: 978-1-139-05996-1. DOI: 10.1017/CBO9781139059961.

[2] S. W. Rienstra and A. Hirschberg, *An Introduction to Acoustics*, ser. IWDE report. Technische Universiteit Eindhoven, 2004.

[3] A. P. Dowling and S. R. Stow, "Acoustic analysis of gas turbine combustors," *Journal of Propulsion and Power*, vol. 19, no. 5, pp. 751–764, Sep. 2003, ISSN: 0748-4658, 1533-3876. DOI: 10.2514/2.6192.

[4] T. Poinsot and D. P. Veynante, *Theoretical and numerical combustion*, third. 2001.

[5] J. G. Aguilar, L. Magri, and M. P. Juniper, "Adjoint-based sensitivity analysis of low-order thermoacoustic networks using a wave-based approach," *Journal of Computational Physics*, vol. 341, pp. 163–181, Jul. 2017, ISSN: 00219991. DOI: 10.1016/j.jcp.2017.04.013.

[6] (Oct. 18, 2016). 17.3: Speed of sound, Physics LibreTexts. Library Catalog: phys.libretexts.org, [Online]. Available: https://phys.libretexts.org/Bookshelves/University_Physics/Book%3A_University_Physics_(OpenStax)/Map%3A_University_Physics_I_-_Mechanics%2C_Sound%2C_Oscillations%2C_and_Waves_(OpenStax)/17%3A_Sound/17.03%3A_Speed_of_Sound (visited on 07/07/2020).

[7] N. Noiray and B. Schuermans, "On the dynamic nature of azimuthal thermoacoustic modes in annular gas turbine combustion chambers," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 469, no. 2151, pp. 20 120 535–20 120 535, Jan. 2, 2013, ISSN: 1364-5021, 1471-2946. DOI: 10.1098/rspa.2012.0535.

[8] K. Balasubramanian and R. I. Sujith, "Thermoacoustic instability in a rijke tube: Non-normality and nonlinearity," *Phys. Fluids*, p. 12, 2008. DOI: https://doi.org/10.1063/1.2895634.

[9] N. Noiray, M. Bothien, and B. Schuermans, "Investigation of azimuthal staging concepts in annular gas turbines," *Combustion Theory and Modelling*, vol. 15, no. 5, pp. 585–606, Oct. 2011, ISSN: 1364-7830, 1741-3559. DOI: 10.1080/13647830.2011.552636.