Personal notes: Integral transforms and related

Yahoo!

This is part of a series of personal notes which may be littered with terrible typographical errors, scattered with gregarious grammatical mistakes or cluttered with catastrophic conceptual blunders.

Contents

1 Introduction

Some examples of integral transforms are: Fourier, Abel, Hilbert, Laplace.
Integral: less noisy... derivative, more sensitive and more noisy

2 Radon transform

From [1]:
Whatever the dimensionality of the space, the Radon transform is a set of 1D projections of some object. (or should we say n-1th dimensional projection?)
In 2D, the Radon transform is the set of all line-integral projections of a 2D function, $f(r)$.
In 3D, the Radon transform is no longer a line integral, but a plane integral. It is used in magnetic resonance imaging etc.

2.1 Radon transform: real-life illustration

A typical problem is: given a set of projections, find the original object. The inverse Radon transform gives the solution.

2.2 Radon transform: mathematics

First, consider the 1D projection of a 2D function $f(x, y)$. The 1D projection, $\lambda(x)$, is defined as the line integrals along a series of lines parallel to the y axis. So for a line $x = p$, the projection along this line is denoted as $\lambda(p)$. The Radon transform is then defined as:

$$\begin{aligned}\lambda(p) &= \int_{-\infty}^{\infty} f(p, y)\, dy \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, f(x, y)\, \delta(p - x)\end{aligned}$$

More generally, one may consider projection along the line $p = \mathbf{r} \cdot \mathbf{n}$, and write it as:

$$\lambda_\phi(p) = \int_\infty d^2r\, f(\mathbf{r})\, \delta(p - \mathbf{r} \cdot \mathbf{n})$$

where the integral runs over the infinite plane, ϕ is the angle between the x axis and the unit vector $\mathbf{n}$. Note that the Dirac delta function in the equation is 1D, i.e. it only performs on one of the two integrals. So, it works only on the dimension along $\mathbf{n}$.

Note: the projection $\lambda_\phi(p)$ can be regarded as:

- a 1D function of p, parametrised by p OR
- a function in a 2D space with polar coordinates p and ϕ OR
- alternatively write it as a single 2D vector in Radon space: $\mathbf{p} = p\mathbf{n}$

where ϕ is its polar angle relative to the x axis, and p is the magnitude of the vector $\mathbf{p}$.

3 Relation of Radon transform to Fourier transform

From [2]:
Consider the Fourier transform on $\mathbb{R}^n$.
... ...
The Fourier transform on $\mathbb{R}^n$ is the 1D Fourier transform of the Radon transform.

4 Relation of Radon transform to Abel transform

See Section 5.2.

5 Abel transform

From [3]:
In combustion research, many non-intrusive diagnostic methods provide line-of-sight projection data of various important parameters in the flame region. Let $p(l)$ be the line-of-sight projection data, and $f(r)$ be the flame property.

5.1 Abel transform: real-life illustration

Example:
OH* chemiluminescence imaging is a non-intrusive diagnostic method. It is an imperfect but common marker for volumetric heat release rate [4]. The captured OH* chemilumiescence image is the integrated line-of-sight of the heat release rate, $f(r)$.

How so?
Consider the intensity image as a 2D matrix of points / pixels. Each pixel has an associated scalar quantity known as the intensity. Its value is obtained from line-integrating the OH* emission values along a line that is perpendicular to the flame plane, as shown in . The final image is thus a 2D matrix of these intensity values, whereby each values is a line-integral. In other words, the image gives the projected or volumetric heat release rate (since it is a summation of infinitesimally small flame planes).

Application
In experiments, one would take OH* images. The aim is: given the projection data (OH* images), how do we obtain the spatial distribution of the associated physical quantity? Well, Abel transform is the solution!

5.2 Abel transform: mathematics

First we consider the relation between the spatial distribution of some physical quantity and its associated projected data.
The spatial (in this case, radial) distribution of the physical quantity, $f(r)$, is related to the line-of-sight projection data, $p(l)$, by the following:

$$p(l) = \int_{-\infty}^{\infty} f(r)\, ds \tag{1}$$

where the spatial orthonormal coordinates are l and s (so they are independent of each other), and the radial direction, r, is defined by:

$$r = \sqrt{l^2 + s^2}$$

and the derivative of the radial vector with respect to one of the orthonormal directions is:

$$\begin{aligned} \frac{dr}{ds} &= \frac{1}{2}\frac{2s}{\sqrt{l^2+s^2}} \\ ds &= \frac{\sqrt{l^2+s^2}}{s}dr \\ &= \frac{r}{s}dr \end{aligned}$$

Next, we consider the relation between the projection and the spatial distribution of a flame property in a plane normal to the streamwise axis of an axisymmetric flame.
Let $f(r)$ be a flame property (function), and $p(l)$ be the associated projection function. $p(l)$ is the line-of-sight integration of $f(r)$.

$$\begin{aligned} p(l) &= \int_{-\infty}^{\infty} f(r)\, ds \\ &= 2\int_{0}^{\infty} f(r)\, ds && \text{(assume symmetry about } s = 0\text{)} \\ &= 2\int_{l}^{\infty} f(r)\frac{r}{s}\, dr && \text{(rmb to change integration limits!)} \\ &= 2\int_{l}^{\infty} \frac{rf(r)}{\sqrt{r^2-l^2}}\, dr \end{aligned}$$

The following equation is the analytical expression for the projection function $p(l)$. Notice that in fact, $p(l)$ is the Radon transform of the distribution function $f(r)$! (see section 4)

$$p(l) = 2\int_{l}^{\infty} \frac{rf(r)}{\sqrt{r^2-l^2}}\, dr \tag{2}$$

The analytical inverse of Eqn. 2 is the Abel transform!

$$f(r) = -\frac{1}{\pi}\int_{r}^{\infty} \frac{p'(l)}{\sqrt{l^2-r^2}}\, dl \tag{3}$$

Some properties of the Abel transform:

- exact
- unique

However, notice that the Abel transform requires the calculation of the derivative of $p(l)$.

5.3 Abel transform: code implementation

6 Fourier analysis

Jean-Baptiste Joseph Fourier (un Français, bien sûr) proposed that an arbitrary, continuous function (heat equation, PDE, ...) can be represented by a trigonometric series. This is known as the Fourier series. Eventually, various generalisation (to complex-valued functions etc.) to Fourier series have been made, and this entire Fourier related studies became known as Fourier analysis. Further generalisation led to the idea of harmonic analysis.

6.1 Fourier series

First, a trigonometric series is defined as a series of the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

Furthermore, it is called a Fourier series if for an integrable function f on an interval of 2π, the terms a_n and b_n have the form:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx \end{aligned}$$

Or more generally for an integrable function f on an interval of length L (introduce scaling factor of $\frac{2\pi}{L}$).

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^{L} f(x) \cos \frac{2\pi nx}{L} \, dx \\ b_n &= \frac{2}{L} \int_0^{L} f(x) \sin \frac{2\pi nx}{L} \, dx \end{aligned}$$

Then, $f(x)$ becomes:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos \frac{2\pi nx}{L} + b_n \sin \frac{2\pi nx}{L} \right)$$

Note that the Fourier series expression is valid only in the interval, $[0, L]$, i.e. within the interval $[0, L]$, $f(x)$ may be expressed using this Fourier series.

However, if $f(x)$ has L-periodicity, then the Fourier series will be sufficient to characterise $f(x)$. From there, we can also conclude that a Fourier series has its limitations: it can be used only for periodic functions or for functions on a bounded (compact) interval.

Next, we consider the derivation of the Fourier coefficients...

Now, we consider the synthesis process, i.e. the reconstruction of the function $f(x)$ from the Fourier series:

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos \frac{2\pi nx}{L} + b_n \sin \frac{2\pi nx}{L} \right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{N} \left(R_n \cos \left(\frac{2\pi nx}{L} + \psi_n \right) \right) \end{aligned}$$

where we applied the trigonometric identity:

$$\begin{aligned}a_n \cos\frac{2\pi nx}{L} + b_n \sin\frac{2\pi nx}{L} &= \sqrt{a_n^2 + b_n^2}\cos\left(\frac{2\pi nx}{L} + \tan^{-1}\left(\frac{b_n}{a_n}\right)\right)\\ &= R_n \cos\left(\frac{2\pi nx}{L} + \psi_n\right)\end{aligned}$$

Now, notice that we may generalise $f(x)$ to a complex-valued function by considering $\cos\theta = \frac{1}{2}(z + z*)$.

Then, we have:

$$R_n \cos\left(\frac{2\pi nx}{L} + \psi_n\right) = \frac{1}{2}e^{i\left(\frac{2\pi nx}{L} - \psi_n\right)} + \frac{1}{2}e^{-i\left(\frac{2\pi nx}{L} - \psi_n\right)}$$

A Fourier series is a periodic function composed of harmonically related sinusoids, combined by a weighted summation. The problem is then: finding the weights of each sinusoidal component. The motivation: we can decompose signals into sums of simple trigonometric functions. The idea is how? We need an operator (or mapping or transform...)

The discrete-time Fourier transform is an example of Fourier series.

Elegance of Fourier series:

- linear (is a summation of trigonometric functions), so superposition principle applies
- complete (covers the entire space of basis vectors, i.e. trigonometric functions of all possible discrete frequencies), since we have: $\sum_{n=0}^{\infty} \cos nx + \sin nx$
- independent basis vectors, since trigonometric functions are orthonormal: $\int_0^T \cos nx \cos mx \, dx = 0$ for $n \neq m$. In that sense, it gives a unique solution!

6.2 Fourier transform

Fourier transform is an extension of Fourier series that results when the period of the represented function is lengthened and allowed to approach infinity.

May be a bit confusing, but the operation itself, and the representation in frequency domain are both called Fourier transform. Various variants exist, and may be confusing. The variants serve different purposes. DTFT, DFT, Fourier series, etc.

The Fourier transform is also referred to as a trigonometric transformation since the complex exponential function can be represented in terms of trigonometric functions.

$$\begin{aligned}e^{-i\omega t} &= \cos(-i\omega t) + \sin(-i\omega t)\\ &= \cos(i\omega t) - \sin(i\omega t)\end{aligned}$$

The Fourier transform $X(f)$ for a continuous time series $x(t)$ is defined as:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt \tag{4}$$

where $-\infty < f < \infty \Rightarrow X(f)$ is continuous over an infinite frequency range. Also, $X(f)$ has dimensions of [amplitude · time] if f has dimension of [amplitude].

The corresponding inverse transform is:

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df \quad (5)$$

where $-\infty < t < \infty$.

talk about normalisation constant (of integrals, samples) fftshift, one-sided, two-sided

link with spectrogram as well... and windowing, etc.., “zero-padding, windowing, filtering, pre-whitening”

6.2.1 Windowing

6.2.2 Zero-padding

7 Power Spectral Density

closely related to Fourier transform.

8 References

[1] H. H. Barrett, “Fundamentals of the radon transform,” in *Mathematics and Computer Science in Medical Imaging*, M. A. Viergever and A. Todd-Pokropek, Eds., Berlin, Heidelberg: Springer Berlin Heidelberg, 1988, pp. 105–125, ISBN: 978-3-642-83308-3 978-3-642-83306-9. DOI: 10.1007/978-3-642-83306-9_4.

[2] R. J. Beerends, “An introduction to the abel transform,” in *Miniconference on Harmonic Analysis*, Centre for Mathematics and its Applications, Mathematical Sciences Institute, The Australian National University, 1987, pp. 21–33.

[3] Z.-G. Yuan, “The filtered abel transform and its application in combustion diagnostics,” p. 16, 2003.

[4] R. Balachandran, B. Ayoola, C. Kaminski, A. Dowling, and E. Mastorakos, “Experimental investigation of the nonlinear response of turbulent premixed flames to imposed inlet velocity oscillations,” *Combustion and Flame*, vol. 143, no. 1, pp. 37–55, Oct. 2005, ISSN: 00102180. DOI: 10.1016/j.combustflame.2005.04.009.